

Relativistic ideal Bose gas in Harmonic traps

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Abstract

Using semiclassical approximation method, Bose-Einstein condensation (BEC) of a relativistic ideal boson gas (RIBG) with and without antibosons in three-dimensional (3-D) harmonic traps is investigated. The BEC transition temperature T_c and the Helmholtz free energy at T_c are calculated. The effect of the rest mass of the boson on the properties of the system is also studied. We find that T_c of RIBG is higher than that of the nonrelativistic approximation. The RIBG with antibosons is also investigated and it is found that the Helmholtz free energy of the system with antibosons at T_c is lower than that of the system without antibosons. It implies that the system with antibosons is more stable.

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I. INTRODUCTION

Bose-Einstein condensation (BEC) has attracted a great deal of attention since its theoretical prediction by Einstein in 1925 [1]. Initially, in the theoretical research on the phenomenon, the relativistic corrections are neglected near the BEC transition temperature T_c [2–4]. Lately, it is found that in the universe some systems consist of bosons of very small rest mass. For example, the rest mass of a pair of neutrinos is about 10^{-30} g in the neutrinos system [5, 6]. For such boson systems, the relativistic effects should be considerable. So particular attention has been given to the behavior of the relativistic boson gas.

The properties of the relativistic ideal boson gas (RIBG) have been discussed in early papers [6–10]. In 1965, Landsberg and Dunning-Davies studied the critical temperature T_c and the specific-heat anomaly at T_c of the RIBG [6, 8]. On the basis of the above works, Beckmann *et al.* investigated the behavior of RIBG with and without mass in all space dimensions (d) [10]. It was found that for the massless gas, the specific-heat has a gap in $d > 2$, while for the massive gas, there was a gap in $d > 4$. Although the research on the BEC in the relativistic system has a long history, a complete treatment using the quantum field theory has not been carried out until 1980s, in which the pair creation of boson-antiboson is considered [11–15]. Haber and Weldon firstly gave the high-temperature expansions for thermodynamic functions of the RIBG by taking into account antibosons [11]. Frota and Singh *et al.* exhibited in detail the thermodynamic properties of d -dimensional system with antibosons [12, 14, 15]. Moreover, the more exact treatments of BEC in RIBG are given [16]. Recently, a relativistic BCS-BEC crossover at zero temperature and finite temperature has been studied by the quantum field theory [17–19].

It is worth noticing that the previous investigations on the properties of the relativistic boson gas were mainly focused on the systems in the absence of external potential. As we know, the external potential is very important to control the characteristics of the Bose gas [20, 21]. Su *et al.* discussed the BEC of RIBG in a general external potential by using classical Hamiltonian [5]. In this paper, we study the behavior of the RIBG with and without antibosons in 3-D harmonic traps using the quantum theory.

The outline of this paper is as follows. In section II, we study the BEC transition temperature and Helmholtz free energy at T_c . Meanwhile, we also investigate the effects of the different values of mass on the BEC of the RIBG. In Sec. III, we introduce antibosons

and discuss the BEC in such a system with antibosons, and Sec. IV is a brief summary.

II. RELATIVISTIC BOSE-EINSTEIN CONDENSATION

Consider a RIBG system of N bosons in 3-D harmonic traps, and the potential function $V(\mathbf{r})$ can be expressed as

$$V(\mathbf{r}) = \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2), \quad (1)$$

where m and \mathbf{r} are the rest mass and coordinate of each boson, respectively. ω_x , ω_y and ω_z are the frequencies of anisotropy harmonic traps. By means of the standard Rayleigh-Schrödinger perturbation [22, 23], the total energy of each boson E_{n_x, n_y, n_z} can be given by

$$E_{n_x, n_y, n_z} = E^{(0)} + E^{(1)}, \quad (2)$$

in which

$$E^{(0)} = 3mc^2 + \hbar \left[n_x \omega_x + n_y \omega_y + n_z \omega_z + \frac{1}{2} (\omega_x + \omega_y + \omega_z) \right], \quad (3)$$

$$E^{(1)} = \frac{3\hbar^2}{16mc^2} \left[n_x^2 \omega_x^2 + n_y^2 \omega_y^2 + n_z^2 \omega_z^2 + n_x \omega_x^2 + n_y \omega_y^2 + n_z \omega_z^2 + \frac{1}{2} (\omega_x^2 + \omega_y^2 + \omega_z^2) \right], \quad (4)$$

where \hbar is the Planck constant, and c is the speed of light. n_x , n_y and n_z are 3-D quantum numbers of the boson energy.

To describe RIBG in grand canonical ensemble, the usual expression for the number of particles N in statistical mechanics is

$$N = \sum_n \rho_n = \sum_{n_x, n_y, n_z} \frac{1}{e^{\beta(E_{n_x, n_y, n_z} - \mu)} - 1}, \quad (5)$$

where $\beta = 1/k_B T$, k_B is the Boltzmann constant, and μ is the chemical potential. ρ_n is the average number of bosons in the state of energy E_{n_x, n_y, n_z} . The number of particles in the ground state N_0 becomes to take on macroscopic values corresponding to the onset of BEC. It is convenient to separate out the lowest energy $E_{0,0,0}$ from the sum Eq. (5), so one can write

$$N - N_0 = \sum_{n_x, n_y, n_z \neq 0} \frac{1}{\exp[\beta(E^{(0)} + E^{(1)} - \mu)] - 1}, \quad (6)$$

when BEC occurs, the chemical potential equals to the energy of the lowest state $E_{0,0,0}$, i.e., $\mu_c \rightarrow 3mc^2 + \hbar(\omega_x + \omega_y + \omega_z)/2 + 3\hbar^2(\omega_x^2 + \omega_y^2 + \omega_z^2)/32mc^2$. Eq. (6) can be reduced as

$$N - N_0 = \sum_{n_x, n_y, n_z \neq 0} \frac{1}{\exp[\beta(\varepsilon^{(0)} + \varepsilon^{(1)})] - 1}, \quad (7)$$

where

$$\varepsilon^{(0)} = \hbar(n_x\omega_x + n_y\omega_y + n_z\omega_z), \quad (8)$$

$$\varepsilon^{(1)} = \frac{3\hbar^2}{16mc^2}(n_x^2\omega_x^2 + n_y^2\omega_y^2 + n_z^2\omega_z^2 + n_x\omega_x^2 + n_y\omega_y^2 + n_z\omega_z^2). \quad (9)$$

In order to evaluate the above sum explicitly, it is assumed that the energy spacing becomes smaller when $N \rightarrow \infty$. So sums (7) become integrals as

$$N - N_0 = \int_0^\infty \frac{dn_x dn_y dn_z}{\exp[\beta(\varepsilon^{(0)} + \varepsilon^{(1)})] - 1}. \quad (10)$$

This approximate transformation is good when trapped particles become very large and $k_B T \gg \hbar\omega_x, \hbar\omega_y, \hbar\omega_z$. So the BEC transition temperature of the RIBG can be obtained by taking $N_0 \rightarrow 0$, one gets

$$N = \int_0^\infty \frac{dn_x dn_y dn_z}{\exp[\beta_c(\varepsilon^{(0)} + \varepsilon^{(1)})] - 1}, \quad (11)$$

where $\beta_c = 1/k_B T_c$. Fig. 1 displays the exact T_c numerically versus N for different values of rest mass. It is found that the BEC transition temperature increases with the decrease of the rest mass of the particle, which is similar to the case of a uniform Bose gas [6]. It is explained that the relativistic effect is obvious for the boson system with very small rest mass. The transition temperature is compared with the result based on the nonrelativistic approximation (see Fig. 2). It is shown that the relativistic effect results in the increasing of the BEC transition temperature, which disagrees with the existing result [5].

The free energy at the critical point can be also investigated. We now discuss the Helmholtz free energy at T_c in harmonic traps. According to the partition function of the system, the Helmholtz free energy can be expressed as

$$F = \mu_c N + k_B T_C \int_0^\infty dn_x dn_y dn_z \{ \ln[1 - \exp[-\beta_c(E_{n_x, n_y, n_z} - \mu_c)]] \}, \quad (12)$$

where β_c is obtained numerically for a given value of N and E_{n_x, n_y, n_z} is given by Eq. (2). Fig. 3 shows the Helmholtz free energy at T_c as a function of N for different values of the rest mass. It can be seen that the smaller the rest mass of boson is, the lower the Helmholtz free energy is, namely, the system is more stable.

III. ANTIBOSON AND BOSE-EINSTEIN CONDENSATION

To our knowledge, at sufficiently high temperature the quantum field theory requires consideration of particle-antiparticle pair production. If \bar{N} is the number of antibosons, the system is governed by the conservation of the number $Q = N - \bar{N}$, rather than of the numbers N and \bar{N} separately. According to Ref [22], $E = \pm E_n$. For bosons we have $E = +E_n$ and for antibosons we have $E = -E_n$. So $N = \sum_n \rho_n$ is replaced by

$$\begin{aligned} Q = N - \bar{N} &= \sum_n (\rho_n - \bar{\rho}_n) \\ &= \sum_{n_x, n_y, n_z} \left[\frac{1}{e^{\beta(E_{n_x, n_y, n_z} - \mu)} - 1} - \frac{1}{e^{\beta(E_{n_x, n_y, n_z} + \mu)} - 1} \right], \end{aligned} \quad (13)$$

where $\bar{\rho}_n$ is the average number of antibosons in the state of energy $-E_{n_x, n_y, n_z}$. Since the number of bosons (antibosons) in various states must be positively defined, we have $|\mu| \leq \mu_c$. Using the similar method as in Sec. II, the BEC critical temperature can be obtained by

$$\begin{aligned} N - \bar{N} &= \int_0^\infty dn_x dn_y dn_z \left[\frac{1}{\exp[\beta_c(\varepsilon^{(0)} + \varepsilon^{(1)})] - 1} - \frac{1}{\exp[\beta_c(\varepsilon^{(0)} + \varepsilon^{(1)} + 2\mu_c)] - 1} \right]. \end{aligned} \quad (14)$$

Fig. 4 gives the behavior of T_c numerically obtained from Eq. (14) for the different values of the rest mass. It can be seen that the behavior of T_c is similar to the case of the system without antibosons. However, comparing to the result of the system without antibosons, we find that T_c of the system with antibosons is higher (see Fig. 5). This means as the decreasing of the temperature, BEC of the system considering antibosons occurs firstly. It

implies that the system with antibosons is more stable, i.e., has a lower Helmholtz free energy at T_c . This will be shown to be the case indeed.

The Helmholtz free energy of the system with antibosons can be also given by

$$F = \mu_c(N - \overline{N}) + k_B T_C \int_0^\infty dn_x dn_y dn_z \{ \ln[1 - \exp[-\beta_c(E_{n_x, n_y, n_z} - \mu_c)]] + \ln[1 - \exp[-\beta_c(E_{n_x, n_y, n_z} + \mu_c)]] \}, \quad (15)$$

where F is just Eq. (12) but with the second log term caused by antibosons. In Fig. 6, we find that for a given value of N the smaller the rest mass is, the more stable the system is. However, it is found that F of the system considering antibosons is always lower than that of the system without antibosons, which proves the above speculation (see Fig. 7).

IV. SUMMARY

In this paper, we have studied the BEC of the relativistic Bose gas with and without antibosons in harmonic traps by the quantum theory. It can be seen that the BEC transition temperature increases with the decrease of the rest mass of the particle. By using the quantum energy spectrum, we find the relativistic BEC transition temperature is higher than that of the nonrelativistic approximation. T_c of the system with antibosons is higher than that of the system without antibosons, and the calculation of the Helmholtz free energy at T_c further reveals that the system with antibosons is more stable.

V. ACKNOWLEDGMENT

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VI. REFERENCES

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Figure Captions

Fig. 1. The BEC transition temperatures T_c s (in units of K) versus the number of particles N without antibosons for different mass and the harmonic frequencies $\omega_x = \omega_y = 10^{17} \text{ Hz}$, $\omega_z = 2 * 10^{17} \text{ Hz}$. The solid, dotted and dashed lines represent the various values of mass $m = 3 * 10^{-34} \text{ kg}$, $3 * 10^{-33} \text{ kg}$ and $3 * 10^{-32} \text{ kg}$, respectively.

Fig. 2. Difference between the BEC transition temperatures T_c s (in units of K) for relativistic and nonrelativistic approximation with the mass $m = 3 * 10^{-32} \text{ kg}$, harmonic frequencies $\omega_x = \omega_y = 10^{17} \text{ Hz}$, $\omega_z = 2 * 10^{17} \text{ Hz}$. The solid and dotted lines stand for the relationship between the transition temperatures T_c s and the number of particles N for nonrelativistic and relativistic cases, respectively.

Fig. 3. The Helmholtz free energy F (in units of μN) versus the number of particles N without antibosons at T_c for different mass and the harmonic frequencies $\omega_x = \omega_y = 10^{17} \text{ Hz}$, $\omega_z = 2 * 10^{17} \text{ Hz}$. The solid, dotted and dashed lines represent the various values of mass $m = 3 * 10^{-32} \text{ kg}$, $1 * 10^{-32} \text{ kg}$ and $7 * 10^{-33} \text{ kg}$, respectively.

Fig. 4. The BEC transition temperatures T_c s (in units of K) versus the number of particles N with antibosons for different mass and the harmonic frequencies $\omega_x = \omega_y = 10^{17} \text{ Hz}$, $\omega_z = 2 * 10^{17} \text{ Hz}$. The solid, dotted and dashed lines represent the various values of mass $m = 3 * 10^{-34} \text{ kg}$, $3 * 10^{-33} \text{ kg}$ and $3 * 10^{-32} \text{ kg}$, respectively.

Fig. 5. Difference between the BEC transition temperatures T_c s (in units of K) with and without antibosons for the mass $m = 3 * 10^{-32} \text{ kg}$, harmonic frequencies $\omega_x = \omega_y = 10^{17} \text{ Hz}$, $\omega_z = 2 * 10^{17} \text{ Hz}$. The solid and dotted lines stand for the relationship between the transition temperatures T_c s and the number of particles N for the cases without and with antibosons, respectively.

Fig. 6. The Helmholtz free energy F (in units of μN) versus the number of particles N with antibosons at T_c for different mass and the harmonic frequencies $\omega_x = \omega_y = 10^{17} \text{ Hz}$, $\omega_z = 2 * 10^{17} \text{ Hz}$. The solid, dotted and dashed lines represent the various values of mass $m = 3 * 10^{-32} \text{ kg}$, $1 * 10^{-32} \text{ kg}$ and $7 * 10^{-33} \text{ kg}$, respectively.

Fig. 7. Difference between the Helmholtz free energy F (in units of μN) with and without antibosons at T_c for the mass $m = 3 * 10^{32} \text{ kg}$, harmonic frequencies $\omega_x = \omega_y = 10^{17} \text{ Hz}$, $\omega_z = 2 * 10^{17} \text{ Hz}$. The solid and dotted lines stand for the relationship between the Helmholtz free energies F and the number of particles N for the cases without and with antibosons, respectively.













